Name:
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Advanced Quantum Mechanics UE
WS 2020/21 Written exam Dec. 9 2020

Start each exercise on a new sheet of paper (+1 point if you comply).
Don't forget to write your name and Matrikelnnumer on each sheet
You are allowed to use an A4 sheet with your own hand-written formulas (no solved exercises), paper, pen.
Cellphones must be switched off and put in your bag or pocket

$\hbar = c = 1$ everywhere

1 Addition of angular momentum (14P)

Two interacting particles $i = 1, 2$ with spin $s_1 = 3/2$ and $s_2 = 1/2$, respectively are described by the hamiltonian

$$\hat{H} = 2\lambda\, \hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2 \qquad \text{(scalar product)} ,$$

where $\hat{\mathbf{S}}_i$ is the spin operator of particle i, and λ a positive constant. The spatial part of the wavefunction is neglected.

(a) Let $\hat{\mathbf{J}} = \hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2$ be the total spin (internal angular momentum). What are the possible (eigen)values of $\hat{\mathbf{J}}^2$ and of the associated quantum number j?

(b) Determine the eigenvalues of $\hat{H}$ as well as their degeneracies.

The basis states of the two-spins system can be expressed, on the one hand, as tensor products

$$|m_1\rangle\, |m_2\rangle \; . \tag{1}$$

Or as $|j, m\rangle$, i.e. eigenstates of $\hat{\mathbf{J}}^2$ and $\hat{J}^z$.

(c) Express the states $|j = 2, m = 2\rangle$ and $|j = 2, m = -2\rangle$ in terms of the product states (1).

(d) Verify that the total number of basis states in the product state (1) and in the $|j, m\rangle$ representation is the same.

the table of Clebsch-Gordan coefficients can be used only for point (g) NOT for (e),(f)

(e) Express the state $|j = 2, m = 1\rangle$ in terms of the product states (1), by applying $\hat{J}^- = \hat{S}_1^- + \hat{S}_2^-$.

(f) Express the state $|j = 1, m = 1\rangle$ in terms of the (1)

(g) Using the table of Clebsch-Gordan coefficients, express the state $|j = 1, m = 0\rangle$ in terms of the (1). What is the probability that a measure of $\hat{S}_1^z$ in this state gives $-\frac{1}{2}$?

$$L^- |\ell, m\rangle =$$
$$= \sqrt{\ell(\ell+1) - m(m-1)}\; |\ell, m-1\rangle$$

3/2×1/2

		2			
		+2	2	1	
+3/2	+1/2	1	+1	+1	
+3/2	−1/2	1/4	3/4	2	1
+1/2	+1/2	3/4	−1/4	0	0
+1/2	−1/2	1/2	1/2	2	1
−1/2	+1/2	1/2	−1/2	−1	−1
−1/2	−1/2	3/4	1/4	2	
−3/2	+1/2	1/4	−3/4	−2	
−3/2	−1/2	1			

−1	−1/2	4/5	1/5	
−2	+1/2	1/5	−4/5	
−2	−1/2			
5/2	3/2	1/2		
1/2	+1/2	+1/2		
/10	2/5	1/2		
3/5	1/15	−1/3	5/2 3/2 1/2	

Please turn!!

2 Two electrons in an harmonic oscillator (14P=4+4+3+3)

Consider two electrons with spin $\frac{1}{2}$ and mass $m = 1$ in a one-dimensional harmonic oscillator with eigenfrequency ω.

(a) Write down the two-particle ground state $|G\rangle$ and the first excited state(s) $|E_i\rangle$ together with their (eigen-)energies and degeneracies. Express the states as tensor products of a a suitably (anti)symmetrized orbital and a spin part.

(b) Same as (a) for the next set of excited states $|F_i\rangle$ (the ones with energy 3ω)

We now introduce the interaction between the two electrons as

$$\hat{V} = \alpha\ \hat{x}_1\hat{x}_2 \equiv \hat{V}_O \otimes \mathbb{I}_S\ ,$$

where $\hat{x}_i$ is the position operator of particle i.

(c) Evaluate all matrix elements of $\hat{V}$ between the states $|E_i\rangle$

(d) Evaluate the first-order correction to the energy of the first excited states within degenerate perturbation theory. What degeneracy is left?

Hints : Just to avoid misunderstanding: no second quantisation here!

$x_i = \frac{1}{\sqrt{2\omega}}\left(b_i + b_i^\dagger\right)$, where b_i and $b_i^\dagger$ are ladder operators for the harmonic oscillator.

For an efficient evaluation of the matrix elements of $\hat{V}_O$ I suggest the following: (i) apply $\hat{V}_O$ to each orbital state and (ii) neglect all terms that fall outside of the degenerate subspace. (iii) Try to write the resulting state in terms of another state in the degenerate subspace.

3 Electromagnetic field (12 P)

Given the hamiltonian for a particle of mass m and charge $q = 1$ in an electromagnetic field

$$H = \frac{1}{2m}\left(p_x + \alpha y\right)^2 + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} \tag{2}$$

(a) Determine the vector potential and the magnetic field

(b) Write down the Heisenberg equations of motion for x and y.

(c) Write down the Heisenberg equations of motion for p_x, p_y.

(d) Use (b,c) to write an equation of motion for v_y and solve this.

(e) Which of the two following ansatzes for the solution of the (time independent) Schrödinger equation is correct and why (we neglect the z-component)?

$$\psi(x,y) = e^{ikx} f(y) \qquad \psi(x,y) = e^{iky} f(x)$$

(f) Write down the resulting equation for the function f.