

Exam "Statistical Physics", Winter Term 2025/26

January 21, 2026

**Problem A** (Independent and distinguishable particles). An isolated system is composed of  $N \gg 1$  independent distinguishable particles. Each one of these particles has only two energy levels  $\varepsilon_1 = 0$  and  $\varepsilon_2 = \varepsilon > 0$ .

- Compute the entropy  $S(E, N)$  of this system.
- State the total energy  $E$  as a function of the temperature  $T$ . Discuss the limits  $T \rightarrow 0$  and  $T \rightarrow +\infty$ .

**Problem B** (Two-dimensional gas). A free electron gas with single-particle energy

$$\varepsilon_{\vec{p}} = \frac{\vec{p}^2}{2m} = \frac{\hbar^2 k^2}{2m},$$

and spin  $s = \frac{1}{2}$  (spin degeneracy  $g_s = 2$ ) is confined to a two-dimensional domain of area  $A$  (with periodic boundary conditions). Let  $n = \langle N \rangle / A$  be the particle density.

- Derive the density of states (DOS) in 2D,  $g(\varepsilon)$ , and also give the DOS *per unit area*,  $g(\varepsilon)/A$ .
- Using your DOS, compute the  $T = 0$  Fermi energy  $\varepsilon_F$  as a function of the density  $n$ .
- At finite temperature, use the grand-canonical expression to derive an *explicit* formula for  $\mu(T)$  in terms of  $T$  and  $\varepsilon_F$ .

**Hint:** You may use the identity

$$\int_0^\infty dx \frac{1}{\alpha e^x + 1} = \ln\left(1 + \frac{1}{\alpha}\right), \quad \alpha > 0.$$

In case that you did not solve (a), assume  $g(\varepsilon) = \text{const}$ .

- Show that  $\mu(T) \rightarrow \varepsilon_F$  as  $T \rightarrow 0$ .

**Problem C** (Degenerate states). A system in contact with a heat reservoir at temperature  $T$  is composed of  $N$  *distinguishable*, independent particles. Each particle can occupy two energy levels:

$$\varepsilon_1 \text{ with degeneracy } g_1, \quad \varepsilon_2 \text{ with degeneracy } g_2, \quad \Delta = \varepsilon_2 - \varepsilon_1 > 0, \quad \beta = \frac{1}{k_B T}.$$

- (a) Compute the *single-particle* partition function  $z(T)$  and the  $N$ -particle partition function  $Z(N, T)$ .
- (b) Compute the probability  $p_2(T)$  that a given particle is in level  $\varepsilon_2$ , and use it to find the mean energy

$$U(N, T) = \langle E \rangle.$$

**Hint:** Express your final result using  $x = e^{-\beta\Delta}$ .

- (c) Compute the heat capacity  $C(N, T) = \left(\frac{\partial U}{\partial T}\right)_N$ .
- (d) Compute the energy variance  $\langle (\Delta E)^2 \rangle$ .