Advanced Quantum Mechanics VO
WS 2022/23 Written exam Feb 2, 2023

Start each exercise on a new sheet of paper. Don't forget to write your name and Matrikelnnumer on the cover
You are allowed to use one A4 sheets, WRITTEN ON ONE SIDE ONLY, with your own hand-written formulas (no solved exercises), Cellphones must be switched off and put in your bag or pocket

1 Quantisation of the Schrödinger field (10P)

A modified lagrangian of the Schrödinger field reads

$$\mathcal{L} = \int \left[i\, a\, \Psi^*\, \dot{\Psi} - i\, a\, \dot{\Psi}^*\, \Psi + b\, \Psi^*\, \nabla^2 \Psi \right] d^3x \,. \tag{1}$$

where a and b are constants.

In the following, you can choose to remove the $-i\, a\, \dot{\Psi}^*\, \Psi$ (the "blue" term) from (1) (state that explicitly at the beginning of the exercise if you do), but then you can achieve a maximum of 6P from this exercise (1P each for a,b,c,d, 2P for e).

(a) Derive one of the two Lagrange II equations for the fields (your choice). Collect explicitly all the terms on one side of the equation.

(b) Write down explicitly the canonical momenta

(c) From the commutation rules between fields and canonical momenta derive the commutation rules [1] between the $\Psi(x)$ and the $\Psi^\dagger(y)$. (Indicate here explicitly the arguments x, y, etc.)

(d) Derive and write down the Hamilton function in terms of the Ψ and the Ψ^*.

(e) Write down the expression of the Hamiltonian in second quantisation for this exercise in terms of annihilation and creation operators b_n and $b_n^\dagger$ for an arbitrary orthonormal basis $\{\chi_n\}$ (i.e. not the eigenbasis). Write explicitly the form of the expression for the matrix elements in terms of an integral over the $\chi_n(x)$.

2 Gauge invariance (8P)

Consider two Hamiltonians describing a charged particle (we consider just one dimension and we take units in which the charge $q = 1$, $c = 1$, $\hbar = 1$):

$$\hat{H}_1 = \frac{1}{2m}\left(\hat{p} + \alpha t^3\right)^2 \qquad \hat{H}_2 = \frac{\hat{p}^2}{2m} - \beta t^2 \hat{x} \tag{2}$$

with α, β constants, t the time.

(a) Write down the Heisenberg equations of motion for the operators $\hat{x}$ and $\hat{p}$ for the two hamiltonians.

(b) Solve them for $\hat{v} = \dot{\hat{x}}$ (velocity) and $\hat{p}$ explicitly.

(c) By comparing the results for $\hat{H}_1$ and $\hat{H}_2$, determine for which value of β the velocity $\hat{v}$ has the same time dependence.

(d) For both Hamiltonians, identify the scalar and vector potentials and determine the electric field.

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[1] If you keep the "blue" term, use the comutation rules between both fields (Ψ and Ψ^*) and their canonical momenta and show that they give the same result.

Please turn over

3 Second quantisation (10P)

Consider an Hamiltonian for bosonic particles, which in second quantisation reads [2]

$$\hat{H} = \alpha \sum_{k=0}^{\infty} (k-2)^2 \, b_k^\dagger b_k \qquad \alpha > 0 \tag{3}$$

(a) Write down the normalized ground state $|G_N\rangle$ with N particles in second quantisation, as well as its energy.
We add now a perturbation $\hat{V}$ which in first quantisation has the form

$$\hat{V}_{FQ} = h \sum_i \hat{q}_i$$

where $\hat{q}$ is an operator with matrix elements between the corresponding levels

$$\langle \varphi_k | \, \hat{q} \, | \varphi_{k'} \rangle = \beta \; k \; \delta_{k,k'}, \qquad \beta > 0 \qquad k = 0, 1, \cdots, \infty$$

(b) Express $\hat{V}$ in second quantisation.
(c) Determine the first-order correction to the energy of $|G_N\rangle$ due to $\hat{V}$.
We now instead add an interaction that in first quantisation reads

$$\hat{W}_{FQ} = \frac{1}{2} \, U \sum_{i \neq j} \, \hat{q}_i \, \hat{q}_j \tag{4}$$

(d) Write down the corresponding interaction $\hat{W}$ in second quantisation.
(e) Determine the corresponding first order correction to the energy of $|G_N\rangle$.

Alternatively, if you are not able to get (b) and/or (d):, you can use for (c) and (e) the generic form for a bosonic single-particle operator in SQ (for $\hat{V}$) and for an interaction in second quantisation (for $\hat{W}$) without determining the coefficients.

[2] Consider carefully where the minimum of $(k-2)^2$ is!