Advanced Quantum Mechanics VO
WS 2020/21 Written exam Feb. 1, 2021

Start each exercise on a new sheet of paper.
You are allowed to use one A4 sheets with your own hand-written formulas (no solved exercises),
$\hbar = 1$ **everywhere**

1. Two-level boson model (8P)

We consider bosonic particles on a two-level system described by the Hamiltonian

$$\hat{H}_0 = -t \left(b_1^\dagger b_2 + b_2^\dagger b_1 \right) \qquad t > 0 \tag{1}$$

where $b_i^\dagger / b_i$ are creation/destruction operators for a particle on level i ($i = 1, 2$).
Solve the hamiltonian in the following way:
Introduce two new creation ($d_A^\dagger$, $d_B^\dagger$) and destruction (d_A, d_B) operators related to the b_i by the transformations

$$b_1 = \alpha \, (d_A + d_B) \qquad b_2 = \alpha \, (d_A + \beta \, d_B) \qquad \alpha, \beta = \text{real constants}, \alpha > 0 \tag{2}$$

and their hermitian conjugates.
(a) Given that the d_x ($x = A, B$) obey the correct commutation relations

$$[d_x, d_y^\dagger] = \delta_{x,y}$$

determine for which values of the constants α, β, the b_i, $b_i^\dagger$ obey correct commutation relations as well.
For simplicity, you just need to do that for the commutators containing $b_2^\dagger$.
(b) Show that in terms of the d_x the Hamiltonian becomes

$$\hat{H}_0 = -p \; d_A^\dagger d_A + q \; d_B^\dagger d_B \qquad p, q > 0 \tag{3}$$

and determine the values of p and q.
(c) Write down the normalized $N = 2$-particle ground state $|G\rangle$ in second quantisation (in terms of the $d_{A/B}^\dagger$) and determine its eigenenergies and degeneracies.
(d) Same for the first excited state $|E\rangle$
Notice: you can solve (c),(d) even without having solved (a),(c). In this case, use directly (3) without determining p, q.

2. Wigner-Eckart's theorem (8P)

Consider a hydrogen atom in a state $|n, \ell, m\rangle$, where $n = 9$ and ℓ, m are the usual angular momentum quantum numbers.
For which values of (ℓ, m) are the following matrix elements nonzero?
[Indicate explicitly the pairs (ℓ, m), example: $(2, 1), (3, 0), (5, -2)$].
(a) $\langle n, \ell, m | \, \hat{z} \, | n, 3, -3\rangle$
(b) $\langle n, \ell, m | \, \hat{x} \, | n, 4, 3\rangle$
(c) $\langle n, \ell, m | \, \hat{x} + i \, \hat{y} \, | n, 4, 0\rangle$
(d) $\langle n, \ell, m | \, (\hat{x}^2 + \hat{y}^2 + \hat{z}^2) \, | n, 2, -1\rangle$

Please turn over

3. Wigner-Eckart's theorem ($j = \frac{3}{2}, \frac{1}{2}$ states) (8P)

Consider now a set of states $|\beta, j, m\rangle$, where β is some quantum number, and j, m are the usual quantum numbers associated to $\mathbf{J}^2$ and $\mathbf{J}_z$. $\hat{\mathbf{V}}$ is a vector operator. Given the matrix element

$$\left\langle \beta, \frac{3}{2}, \frac{1}{2} \middle| \hat{V}_Z \middle| \beta, \frac{1}{2}, \frac{1}{2} \right\rangle = \alpha$$

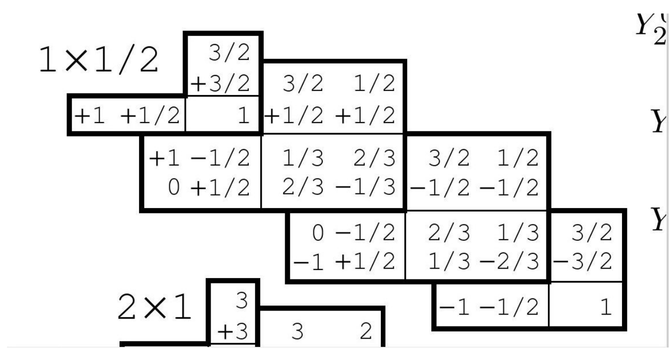
determine the following matrix elements in terms of the real quantity α, whenever possible and indicate when this is not possible (i.e. when the knowledge of α is not sufficient).

(a) $\left\langle \beta, \frac{3}{2}, m \middle| \hat{V}_Z \middle| \beta, \frac{1}{2}, -\frac{1}{2} \right\rangle$

(b) $\left\langle \beta, \frac{1}{2}, m \middle| \hat{V}_Z \middle| \beta, \frac{3}{2}, \frac{1}{2} \right\rangle$

(c) $\left\langle \beta, \frac{1}{2}, \frac{1}{2} \middle| \hat{V}_Z \middle| \beta, \frac{1}{2}, \frac{1}{2} \right\rangle$

(d) $\left\langle \beta, \frac{3}{2}, m \middle| \hat{V}_X \middle| \beta, \frac{1}{2}, \frac{1}{2} \right\rangle$



Notice: The parity selection rule does not apply here.
Hint: $\hat{V}_{\pm 1} = \mp(\hat{V}_X \pm i\hat{V}_Y)/\sqrt{2}$

4. Fermi Sphere (8P)

The energy of a free electron with momentum $\mathbf{k}$ is given by $\varepsilon_{\mathbf{k}} = \frac{\hbar^2 \mathbf{k}^2}{2m}$.
$c^\dagger_{\mathbf{k},\sigma}$, $c_{\mathbf{k},\sigma}$ are creation and destruction operators for an electron with momentum $\mathbf{k}$ and spin σ. The momenta are discrete.
(a, 2P) Write down the corresponding Hamiltonian in second quantisation (SQ).
(b) Describe in words its ground state $|F\rangle$ (i.e.: what is the filled Fermi sphere, which levels are occupied?).
(c) Write down the ground state $|F\rangle$ in SQ
(d) Write down the expression for the total number of particles N_p and the total energy E_T of $|F\rangle$. (Their values, not operators. It's a sum over $\mathbf{k}, \sigma$, you don't need to transform it into an integral, nor to evaluate it. However, indicate the range of $\mathbf{k}$)
(e) Write down in SQ the operator counting the number of particles with momentum $\mathbf{k}$ and spin σ.
(f) Write down an excited state (particle-hole excitation) in SQ, as well as its excitation energy (i.e. the difference between its energy and E_T). You just have to apply suitable operators to $|F\rangle$
(g) Explain (f) by using the Fermi sphere.