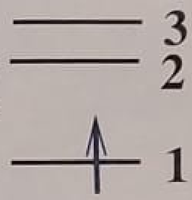
Advanced Quantum Mechanics VO
WS 2024/25 Written exam July 17, 2025

Don't forget to write your name and Matrikelnnumer
Cellphones must be switched off and put in your bag or pocket
You can use $\hbar = 1$ and $c = 1$ everywhere

1 Bosons in second quantisation (12P)

Consider bosonic particles whose first quantisation Hamiltonian $\hat{H}_{FQ}$ has eingenstates (levels) $|\varphi_k\rangle$ and eigenenergies $\varepsilon_k = -\alpha/k$, with $k = 1, 2, \cdots, \infty$.
Here, α is a positive constant. **Hint:** Notice the minus sign. The figure may help. (For the energies you can approximate $\frac{1}{2} \approx 0.5$ and $\frac{1}{3} \approx 0.3$).
Let $b_k^\dagger, b_k$ be the annihilation and creation operators associated with the orbital level $|\varphi_k\rangle$



(a) write down the Hamiltonian $\hat{H}$ in second quantisation
(b) write down the normalized ground state(s) $|G_N\rangle$ of $\hat{H}$ with N particles in second quantisation and determine their eigenenergies and degeneracies.
(c) same as (b) for the first excited state(s) $|E_N\rangle$.
(d) same as (b) for the second excited states $|S_N\rangle$.
(e) We add now a perturbation $\hat{V}$ which in second quantisation reads

$$\hat{V} = h \sum_{k=1}^{\infty} \sum_{p=0,1} \left(b_k^\dagger b_{k+p} + b_{k+p}^\dagger b_k \right)$$

Determine the first order correction to the energy of $|G_N\rangle$
(f) Determine the second order correction to the energy of $|G_N\rangle$

2 Scattering within the first-order Born approximation (10P)

Particles of mass m and momentum $\mathbf{k}$ are scattered by a potential

$$V(r) = \frac{u}{a^2}\,\delta(r-a) \qquad r \equiv |\mathbf{x}|\,.$$

Within the first-order Born approximation:
(a) Determine the scattering amplitude $f(\mathbf{k}, \mathbf{k}_{out})$.
(b) Determine the differential cross section $d\sigma/d\Omega$ as a function of the scattering angle θ
(c) Write down the expression for the total cross section σ in terms of an integral over θ. You don't need to carry out this integral.
(d) Determine σ for the case of low energies $ka \ll 1$ (in this case, carry out the integral).
(e) Be ρ the particle density. Determine how many particle per unit time are scattered between the angles θ_1 and θ_2 (not necessarily small angles).

Please turn over

3. Magnetic field (12P)

Given the hamiltonian for a particle of mass m and charge $q = 1$ in a magnetic field

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{(p_z + \alpha x)^2}{2m} \tag{1}$$

(a) Write down the Heisenberg equations of motions for p_x, p_z and x.
(b) Use (a) to write a differential equation for v_x and solve it.
(c) Which one of the following ansatzes for the solution of the (time independent) Schrödinger equation is correct and why?

$$\psi(x,y,z) = e^{i\ k_2\ y + i\ k_3\ z} f(x) \qquad \psi(x,y,z) = e^{i\ k_2\ y + i\ k_1\ x} f(z)$$

$$\psi(x,y,z) = e^{i\ k_2\ y + i\ k_1\ x + i\ k_3\ z}\ f(z) \qquad \psi(x,y,z) = e^{i\ k_1\ x} f(y,z)$$

(d) Write down the resulting Schrödinger equation for the function f.
(e) Introduce the appropriate shift of one coordinate that makes it equivalent to the Schrödinger equation for an harmonic oscillator. Determine the frequency ω of this oscillator.
Hint: $H_{harm.osc.} = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2$
(f) Determine the energy eigenvalues of the system. They don't depend on a particular quantum number, which one?