Advanced Quantum Mechanics UE
WS 2024/25 Written exam Jan. 10 2025

Start each exercise on a new sheet of paper. Don't forget to write your name and Matrikelnnumer on each sheet
You are allowed to use one A4 sheet with your own hand-written formulas, written on one side only, (no solved exercises),
Cellphones must be switched off and put in your bag or pocket

$\hbar = c = 1$ **everywhere**

1 Addition of angular momentum (14P)

Two interacting particles $i = 1, 2$ with spin $s_1 = 3$ and $s_2 = 1$, respectively are described by the hamiltonian

$$\hat{H} = 2\lambda\, \hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2 \qquad \text{(scalar product)} ,$$

where $\hat{\mathbf{S}}_i$ is the spin operator of particle i, and λ a positive constant. The spatial part of the wavefunction is neglected.
(a) Let $\hat{\mathbf{J}} = \hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2$ be the total spin (internal angular momentum). What are the possible (eigen)values of $\hat{\mathbf{J}}^2$ and of the associated quantum number j?
(b) Determine the eigenvalues of $\hat{H}$ as well as their degeneracies.
The basis states of the two-spins system can be expressed, on the one hand, as tensor products

$$|m_1\rangle\, |m_2\rangle \ . \tag{1}$$

Or as $|j, m\rangle$, i.e. eigenstates of $\hat{\mathbf{J}}^2$ and $\hat{J}^z$.
(c) Verify that the total number of basis states in the product state (1) and in the $|j, m\rangle$ representation is the same.
(d) Express the states $|j = 4, m = 4\rangle$ and $|j = 4, m = -4\rangle$ in terms of the product states (1).
(e) Express the state[1] $|j = 4, m = 3\rangle$ in terms of the product states (1), by applying $\hat{J}^- = \hat{S}_1^- + \hat{S}_2^-$.
(f) Express the state $|j = 3, m = 3\rangle$ in terms of the (1) (**Hint:** orthogonalisation)
(g) What is the probability that a measure of $\hat{S}_2^z$ in this state gives 0 ?

Hint: $J^- |\ell, m\rangle = \sqrt{\ell(\ell + 1) - m(m - 1)}\ |\ell, m - 1\rangle$

2 Electric field and Gauge invariance (10P)

We consider a particle with charge $q = 1$ and mass $m = 1$ in an homogeneous, time-dependent electric field

$$\mathbf{E} = 2\mathcal{E}\mathbf{e}_x\, t \tag{2}$$

[1] No normalisation is requested for the states here and below

Please turn!!

We can restrict the problem to one dimension, i.e. consider just the x coordinate, and use units in which $\hbar = c = 1$.
(a) Different forms of the vector ($\mathbf{A}$) and the scalar (φ) potentials related by a gauge transformation give the same $\mathbf{E}$. Find $\mathbf{A}(x,t)$ and $\varphi(x,t)$ for the following two cases (gauges. One particular solution each is sufficient):
(1) $\mathbf{A}_1(x,t) \neq 0$, $\qquad \varphi_1(x,t) = 0$
(2) $\mathbf{A}_2(x,t) = 0$, $\qquad \varphi_2(x,t) \neq 0$.
(b) Find also the gauge transformation (i.e. one possible choice for the function $\chi(x,t)$ generating it) r which transforms the potentials of (1) into the ones of (2). Write down the Hamiltonian H_1 for (1) and H_2 for (2).
Hint: Ansatz: (1)$\mathbf{A}_1 = f_1(t)$, (2) $\varphi_2 = f_2(t)x$, $\quad \chi = f_3(t)\ x$,
(c) Write down the Heisenberg equations of motion for the operators x and p of the charged particle discussed above for both gauges. (Provide the result: it is not sufficient to just write the commutators).
(d) Solve them for $v = \dot{x}$ (velocity) and p explicitly.
(e) Is v gauge invariant, is p gauge invariant? Argue based on the results from (d).

3 Bosons in second quantization (8P)

Consider bosonic particles whose first quantisation Hamiltonian $\hat{H}_{FQ}$ has eigenstates (levels) $|\varphi_k\rangle$ and eigenenergies $\varepsilon_k = \alpha \cdot k$, with $k = 1, 2, \cdots, \infty$ and α a positive constant (please, notice the values of k).
Let $b_k^\dagger, b_k$ be the creation and destruction operators for a particle in the level $|\varphi_k\rangle$.
(a) Write down the corresponding Hamiltonian $\hat{H}$ in second quantisation

For a given (generic) value of the total number of particles N ($N \geq 2$)
(b) write down the normalized ground state $|G_N\rangle$ of $\hat{H}$ in second quantisation and determine its eigenenergy and degeneracy.
(c) same as (b) for the first excited state(s) $|E_N\rangle$.
(d) Same as (b) for the second excited state(s) $|S_N\rangle$.

4 Wigner-Eckart's theorem (8P)

Consider a hydrogen atom in a state $|n, \ell, m\rangle$, where $n = 6$ and ℓ, m are the usual angular momentum quantum numbers.
For which values of (ℓ, m) are the following matrix elements nonzero?
[Indicate explicitly the pairs (ℓ, m), example: $(2,1), (3,0), (5,-2)$].
(a) $\langle n, \ell, m|\,(2\hat{x}^2 + 2\hat{y}^2 + 2\hat{z}^2 - 2\ \hat{\mathbf{r}} \cdot \hat{\mathbf{p}})\,|n, 4, -2\rangle$
(b) $\langle n, \ell, m|\,\hat{z}\,|n, 3, -3\rangle$
(c) $\langle n, \ell, m|\,\hat{y}\,|n, 4, -4\rangle$
(d) $\langle n, \ell, m|\,\hat{x} - i\ \hat{y}\,|n, 2, 1\rangle$
Hint: $\hat{V}_{\pm 1} = \mp(\hat{V}_x \pm i\hat{V}_y)/\sqrt{2}$