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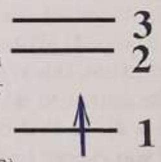
Advanced Quantum Mechanics VO
WS 2024/25 Written exam Feb 4, 2025

Start each exercise on a new sheet of paper.
Don't forget to write your name and Matrikelnnumer at the beginning of each exercise and on the cover.
Cellphones must be switched off and put in your bag or pocket
You can use $\hbar = 1$ everywhere

1 Fermions in second quantisation (12P)

Consider fermionic particles whose first quantisation Hamiltonian $\hat{H}_{FQ}$ has eingenstates (levels) $|\varphi_k\rangle$ and eigenenergies $\varepsilon_k = -\alpha/k$, with $k = 1, 2, \cdots, \infty$. Here, α is a positive constant. **Hint:** Notice the minus sign. The figure may help. (For the energies you can approximate $\frac{1}{2} \approx 0.5$ and $\frac{1}{3} \approx 0.3$).

Let $c^\dagger_{k,\sigma}, c_{k,\sigma}$ be the annihilation and creation operators associated with the orbital level $|\varphi_k\rangle$ with spin index $\sigma = \pm 1$ (or $\sigma = \uparrow, \downarrow$).



(a) write down the Hamiltonian $\hat{H}$ in second quantisation

(b) write down the normalized ground state(s) $|G_N\rangle$ of $\hat{H}$ with $N = 1, 2, 3$ particles in second quantisation and determine their eigenenergies and degeneracies.

(c) same as (b) for the first excited state(s) $|E_2\rangle$ (i.e. with $N = 2$).

(d) same as (b) for the second excited states $|S_2\rangle$.

(e) We add now a perturbation $\hat{V}$ which in second quantisation reads

$$\hat{V} = h \sum_{k=1}^{\infty} \sum_{p=0,1} \sum_{\sigma} \left(c^\dagger_{k,\sigma} c_{k+p,\sigma} + c^\dagger_{k+p,\sigma} c_{k,\sigma} \right)$$

Determine the first order correction to the energy of $|G_2\rangle$

(f) Determine the second order correction to the energy of $|G_2\rangle$

2. Scattering within the first Born approximation (8P)

Particles of mass m and momentum $\mathbf{k}$ are scattered by a potential

$$V(\mathbf{r}) = \begin{cases} \frac{\beta}{r}, & \text{for } r < a \\ 0, & \text{for } r > a \end{cases}$$

Within the first-order Born approximation:

(a) Determine the scattering amplitude $f(\mathbf{k}, \mathbf{k}_{out})$ (you must carry out the radial integral).

(b) Determine the differential cross section. Write it explicitly as a function of the scattering angle θ.

We now consider the case of small energy:

(c) Determine the differential and the total cross section in this case.

(d) Be ρ the particle density. Determine how many particle per unit time are scattered through an angle **larger** than θ.

Hint: $\cos\epsilon \approx 1 - \varepsilon^2/2 + \cdots$ $\int \sin x \, dx = -\cos x$

Please turn over

3. Helium (10P)

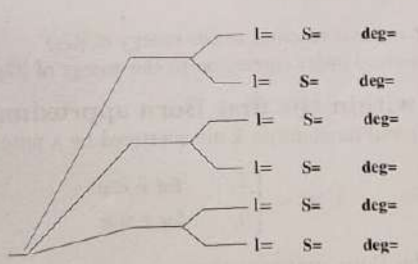
Information: For (a),(b),(c) neglect the electron-electron interaction. Express energies in atomic units.
Express all states as tensor products of a a suitably (anti)symmetrized orbital (= spatial) and a spin part (i.e., not in second quantisation).
Example:

$$\frac{1}{\sqrt{2}}\left(|n_1 l_1 m_1\rangle|n_2 l_2 m\rangle + \cdots\right)|s, m_s\rangle \qquad m = \cdots, m_s = \cdots, \cdots \tag{1}$$

Hint: Energy of an electron in the Coulomb field of a nucleus with charge Z and principal quantum number n.

$$\mathcal{E} = -\frac{1}{2}\frac{Z^2}{n^2}$$

(a) Write down the ground state(s) of Helium, its degeneracy and energy.
(b) Consider the excited state(s) in which one electron has $n = 1$ and the other has $n = 3$. Write down the energy and total degeneracy of these states.
(**Notice,** this is not the first excited state, as we did in class. If you wish, you have the option to do this exercise as we did in class (i.e. with the second electron in $n = 2$). In that case, however, the points from b,c,d will be halved. Write clearly, if you choose to do so.)
(c) Write down the states introduced in (b) using the form Eq. 1. You just need to write the states with the maximum l.
(d) If you introduce electron electron interaction, the degeneracy of the states in (b) is lifted according to the following scheme. Fill in the values of the total angular momentum l, total spin S and degeneracy.



(e) What kind of interaction is responsible for the energy splitting of states with different S?.