Uni Graz & TU Graz
Functional analysis
WS 2025/26

Gernot Eichmann
Adrià Canós Valero
Maximilian Schuster

Exam

Please write your name on every sheet!

Problem 1: Consider the linearly independent system of monomials $\{1, x, x^2, x^3, \dots\}$. Use Gram-Schmidt orthogonalization to construct the first three orthogonal polynomials with respect to the scalar product

$$\langle f, g\rangle = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx\, e^{-x^2} f(x)\, g(x)\,. \tag{1}$$

The following integral (and its derivatives) will be useful:

$$I(\alpha) = \int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}\,, \qquad \alpha > 0\,. \tag{2}$$

Problem 2: Consider the function $f(x) = 1 - |x|$ on the interval $x \in (-\pi, \pi]$. Determine its Fourier series. Does the Fourier series converge uniformly or not, and why?

Problem 3: Use a Laplace transform to solve the following coupled system of integral equations with given initial conditions:

$$\begin{aligned} y' + z' + z &= 0\,, & y(0) = y'(0) &= 0\,, \\ y'' + z' &= 0\,, & z(0) &= 2\,. \end{aligned} \tag{3}$$

Problem 4: Calculate the following integrals:

$$\pi \int_{-\varepsilon}^{\infty} dx\, z^x\, \delta(\sin(\pi x))\,, \qquad \pi \int_{-\varepsilon}^{\infty} dx\, \frac{z^x\, \delta(\sin(\pi x))}{\Gamma(x+1)}\,, \tag{4}$$

where $|z| < 1$ and $0 < \varepsilon < 1$. Remember that $\Gamma(n+1) = n!$ for $n \in \mathbb{N}_0$.

Problem 5: Consider the operator $A : \mathcal{H} \to \mathcal{H}$, $A = -\frac{d^2}{dx^2}$ on the Hilbert space $\mathcal{H} = L^2(0,1)$ with domain

$$\Big\{ f \in \mathcal{H} \,|\, f, f' \text{ absolutely continuous},\ Af \in \mathcal{H},\ f(0) = f(1)\,,\ f'(0) = \alpha\, f'(1) \Big\}\,, \tag{5}$$

where α is some complex number. For which α does A become self-adjoint? Determine the possible eigenvalues of A for that value of α.